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# Towards a theory of chaos explained as travel on Riemann surfaces

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## Abstract

We investigate the dynamics defined by the following set of three coupled first-order ODEs:

$$\dot{z}_n + i\omega z_n = \frac{g_{n+2}}{z_n - z_{n+1}} + \frac{g_{n+1}}{z_n - z_{n+2}}.$$

It is shown that the system can be reduced to quadratures which can be expressed in terms of elementary functions. Despite the integrable character of the model, the general solution is a multiple-valued function of time (considered as a complex variable), and we investigate the position and nature of its branch points. In the semi-symmetric case ( $g_1 = g_2 \neq g_3$ ), for rational values of the coupling constants the system is *isochronous* and explicit formulae for the period of the solutions can be given. For irrational values, the motions are confined but feature aperiodic motion with sensitive dependence on initial conditions. The system shows a rich dynamical behaviour that can be understood in quantitative detail since a global description of the Riemann surface associated with the solutions can be achieved. The details of the description of the Riemann surface are postponed to a forthcoming publication. This toy model is meant to provide a paradigmatic first step towards understanding a certain novel kind of *chaotic* behaviour.

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## 1. Introduction

The fact that the distinction among *integrable* or *nonintegrable* behaviours of a dynamical system is somehow connected with the *analytic structure* of the solutions of the model under

consideration as functions of the independent variable ‘time’ (considered as a *complex* variable) is by no means a novel notion. It goes back to classical work by Carl Jacobi, Henri Poincaré, Sophia Kowalevskaya, Paul Painlevé and others. In recent times some of us had the good fortune to hear in several occasions such ideas clearly described by Kruskal [19, 20]. A simple-minded rendition of his teachings can be described as follows: for an evolution to be *integrable*, it should be expressible, at least in principle, via formulae that are *not excessively multivalued* in terms of the dependent variable, entailing that, to the extent this evolution is expressible by *analytic functions* of the dependent variable (considered as a *complex* variable), it might possess branch points, but it should *not* feature an *infinity* of them that is *dense* in the *complex* plane of the independent variable.

A number of techniques collectively known as Painlevé analysis have been resurrected and further developed over the last few decades (for a review see, for instance, [23]). In essence, they consider an *ansatz* of the local behaviour of a solution near a singularity in terms of a Laurent series, introducing it in the equations and determining the leading orders and resonances (terms in the expansion at which arbitrary constants appear). Painlevé analysis has been extended to test for the presence of algebraic branching (weak Painlevé property [23]) by considering a Puiseux series instead of a Laurent series. These analytic techniques (which have been algorithmized and are now available in computer packages) constitute a useful tool in the investigation of integrability: in many nonlinear systems where no solution in the closed form is known, Painlevé analysis provides information on the type of branching featured by the general solution or by special classes of solutions. It has also proved useful to identify special values of the parameters for which generally *chaotic* systems such as Hénon–Heiles or Lorenz are *integrable* [4, 14].

On the opposite side of the spectrum lie chaotic dynamical systems and it is natural to investigate the singularity structure of their solutions. Tabor and his collaborators initiated this study in the early 1980s for the Lorenz system [24] and the Henon–Heiles Hamiltonian [14]. They realized that the singularities of the solutions in complex time are important for the real-time evolution of the system. The complex time analytic structure was studied by extensions of the Painlevé analysis involving the introduction of logarithmic terms in the expansion—the so-called  $\Psi$ -series—which provides a local representation of the solutions in the neighbourhood of a singularity in the chaotic regime. Their local analytic approach was complemented by numerical techniques developed for finding the location of the singularities in complex time and determining the order of branching [12]. In all the chaotic systems under study, they observed numerically that the singularities in the complex time cluster on a natural boundary with self-similar structure [13]. An analytic argument to explain the mechanism that leads to recursive singularity clustering was given in [21]. Similar studies relating singularity structure, chaos and integrability have been performed by Bountis and his collaborators. Going beyond the local techniques described above, the emphasis is put on a global property of the solutions: whether their Riemann surface has a finite or an infinite number of sheets. Bountis proposes to use the term *integrable* for the first case and *non-integrable* for the second [2, 3]. Using mostly numerical evidence he conjectures that in the non-integrable cases the Riemann surfaces are infinitely sheeted and the projection on the complex plane of the singularities is dense. Combining analytical and numerical results for a simple ODE, Bountis and Fokas [16] have identified chaotic systems with the property that the singularities of their solutions are dense.

Painlevé analysis and its extensions are useful and widely applicable. However, as local techniques, they provide no information on the *global* properties of the Riemann surfaces of the solutions, such as the number and location of the movable branch points a solution has, and moreover how the different sheets of the Riemann surface are connected together at

those branch points. Understanding these global properties is important for the dynamics; a detailed analysis of the Riemann surface associated with the solutions of a dynamical system, whenever it can be done, provides a much deeper understanding than can be obtained by local techniques alone.

This is precisely the motivation of the investigation reported herein: to introduce and study a model which is simple enough that a full description of its Riemann surface can be performed, yet complicated enough to feature a rich behaviour, possibly including irregular or chaotic characteristics.

Such a model was initially presented in [1] and in this paper we continue investigating its properties. This line of research originates from a ‘trick’ that is convenient to identify isochronous systems [5, 7]—a change of dependent and independent variables, with the new independent variable travelling on a path in the complex plane. Later it was shown that many isochronous systems can be written by a suitable modification of a large class of complex ODEs [7, 8]. Using local analysis and numerical integration in two many-body systems in the plane [9, 11], it was discovered that outside the isochrony region there exist periodic solutions with much higher periods as well as possibly aperiodic solutions, and the connection among this phenomenology and the analytic structure of the corresponding solutions as functions of complex time was illuminated. However, those systems were too complicated for a complete description of the Riemann surface to be achieved.

Recent work along these lines includes problems whose solution is obtained by inversion of a hyperelliptic integral: the corresponding Riemann surfaces have been studied in [15, 17], together with the implications on the dynamical properties of the models.

In the present paper we provide many details that were reported without proof in [1], such as the description of the general solution by quadratures, and we also exhibit other properties of the model that were not present in [1], such as similarity solutions, equilibrium configurations and small oscillations. Our investigation of this model will continue in a subsequent publication, [10], where the full description of the geometrical properties of the Riemann surface will be given.

This paper is organized as follows: in section 2 we present our model, including in particular the relationship among its *physical* version (independent variable: the *real* time  $t$ ) and its *auxiliary* version (independent variable: a *complex* variable  $\tau$ ), and we outline the main findings reported in this paper. In section 3 we discuss the equilibrium configurations of our *physical* model, and the behaviour of this system in the neighbourhood of these solutions, and we also obtain certain *exact* similarity solutions of our model and discuss their stability. In section 4 we discuss the analytic structure of the solutions of the *auxiliary* model via local analyses *à la Painlevé*, since the analytic structure of these solutions plays a crucial role in determining the time evolution of our *physical* model. In section 5 we show how the *general solution* of our model can be achieved by *quadratures* and in section 6 we outline the behaviour of our model based on these results. Finally, in section 7 we summarize our results and comment on future developments. This paper also contains a few appendices, where certain calculations are confined (to avoid interrupting inconveniently the flow of the presentation) as well as certain additional findings.

## 2. Presentation of the model

In this section we introduce the model treated in this paper, and we outline our main findings that are then proven and further discussed in subsequent sections.

### 2.1. The auxiliary model

The *auxiliary* model on which we focus in this paper is characterized by the following system of *three* coupled nonlinear ODEs:

$$\zeta_n' = \frac{g_{n+2}}{\zeta_n - \zeta_{n+1}} + \frac{g_{n+1}}{\zeta_n - \zeta_{n+2}}. \quad (1)$$

*Notation.* Here and hereafter indices such as  $n, m$  range from 1 to 3 and are defined mod(3);  $\tau$  is the (*complex*) independent variable of this *auxiliary* model; the *three* functions  $\zeta_n \equiv \zeta_n(\tau)$  are the dependent variables of this *auxiliary* model, and we assume them to be as well *complex*; an appended prime always denotes differentiation with respect to the argument of the functions it is appended to (here, of course, with respect to the *complex* variable  $\tau$ ); and the *three* quantities  $g_n$  are arbitrary ‘coupling constants’ (possibly also *complex*; but in this paper we restrict consideration mainly to the case with *real* coupling constants; this is in particular hereafter assumed in this section). In the following we will often focus on the ‘semisymmetrical case’ characterized by the equality of *two* of the *three* coupling constants, say

$$g_1 = g_2 = g, \quad g_3 = f, \quad (2)$$

since in this case the treatment is simpler yet still adequate to exhibit most aspects of the phenomenology we are interested in. More special cases are the ‘fully symmetrical’, or ‘integrable’, one characterized by the equality of *all three* coupling constants,

$$g = f, \quad g_1 = g_2 = g_3 = g, \quad (3)$$

and the ‘two-body’ one, with only one nonvanishing coupling constant, say

$$g_1 = g_2 = g = 0, \quad g_3 = f \neq 0. \quad (4a)$$

In this latter case clearly

$$\zeta_3' = 0, \quad \zeta_3(\tau) = \zeta_3(0) \quad (4b)$$

(see (1)) and the remaining *two-body* problem is trivially solvable,

$$\zeta_s(\tau) = \frac{1}{2}[\zeta_1(0) + \zeta_2(0)] - (-)^s \left\{ \frac{1}{4} [\zeta_1(0) - \zeta_2(0)]^2 + f\tau \right\}^{1/2}, \quad s = 1, 2, \quad (4c)$$

while the justification for labelling the fully symmetrical case (3) as ‘integrable’ will be clear from the following (or see section 2.3.4.1 of [6]). Before introducing our *physical* model, let us note that the *auxiliary* system (1) is *invariant* under translations of both the independent variable  $\tau$  (indeed, it is *autonomous*) and the dependent variables  $\zeta_n(\tau)$  ( $\zeta_n(\tau) \Rightarrow \zeta_n(\tau) + \zeta_0, \zeta_0' = 0$ ), and it is moreover *invariant* under an appropriate simultaneous rescaling of the independent and the dependent variables.

### 2.2. The trick and the physical model

The trick mentioned above, relating the *auxiliary* model to the *physical* model, amounts in our present case to the introduction of the (*real*) independent variable  $t$  (‘physical time’), as well as the *three* (*complex*) dependent variables  $z_n \equiv z_n(t)$ , via the following positions:

$$\tau = \frac{\exp(2i\omega t) - 1}{2i\omega}, \quad (5a)$$

$$z_n(t) = \exp(-i\omega t)\zeta_n(\tau). \quad (5b)$$

We hereafter assume the constant  $\omega$  to be *real* (for definiteness, *positive*,  $\omega > 0$ ; note that for  $\omega = 0$  the change of variables disappears), and we associate with it the period

$$T = \frac{\pi}{\omega}. \tag{5c}$$

Note that this change of variables entails that the initial values  $z_n(0)$  of the ‘particle coordinates’  $z_n(t)$  coincide with the initial values  $\zeta_n(0)$  of the dependent variables of the *auxiliary* model (1):

$$z_n(0) = \zeta_n(0). \tag{6}$$

It is easily seen that, via this change of variables, (5), the equations of motion (1) satisfied by the quantities  $\zeta_n(\tau)$  entail the following (*autonomous*) equations of motion (in the *real* time  $t$ ) for the particle coordinates  $z_n(t)$ :

$$\dot{z}_n + i\omega z_n = \frac{g_{n+2}}{z_n - z_{n+1}} + \frac{g_{n+1}}{z_n - z_{n+2}}. \tag{7}$$

Here and hereafter superimposed dots indicate differentiations with respect to the time  $t$ .

So, this model (7) describes the ‘physical evolution’ which we study. Note that its equations of motion, (7), are of *Aristotelian*, rather than *Newtonian*, type: the ‘velocities’  $\dot{z}_n$ , rather than the ‘accelerations’  $\ddot{z}_n$ , of the moving particles are determined by the ‘forces’.

Let us immediately emphasize two important *qualitative* aspects of the dynamics of our *physical* model (7). The ‘one-body force’ represented by the second term,  $i\omega z_n$ , on the left-hand side of the equations of motion (7) becomes dominant with respect to the ‘two-body forces’ appearing on the right-hand side in determining the dynamics whenever the (*complex*) coordinate  $z_n$  of the  $n$ th particle becomes large (in modulus). Hence when  $|z_n(t)|$  is very large, the solution  $z_n(t)$  of (7) is approximated by the solution of  $\dot{z}_n + i\omega z_n \approx 0$  implying that  $z_n(t)$  is characterized by the behaviour  $z_n(t) \approx c \exp(-i\omega t)$ , therefore the trajectory of the  $n$ th particle tends to rotate (clockwise, with period  $2T$ ) on a (large) circle. This effect causes *all* motions of our *physical* model, (7), to be *confined*. Second, it is clear that the *two-body* forces (see the right-hand side of (7)) cause a *singularity* whenever there is a *collision* of *two* (or all *three*) of the particles as they move in the *complex*  $z$ -plane, and become dominant whenever *two* or *three* particles get very close to each other, namely in the case of *near collisions*. But if the *three* particles move *aperiodically* in a *confined* region (near the origin) of the complex  $z$ -plane, a lot of *near collisions* shall indeed occur. And since the outcome of a *near collision* is likely to be quite different depending on which side two particles scatter past each other—and this, especially in the case of very close *near collisions*, depends *sensitively* on the initial data of the trajectory under consideration—we see here a mechanism complicating the motion, indeed causing some kind of *chaos* associated with a *sensitive dependence* of the motion on its initial data. This suggests that our model (7), in spite of its simplicity, is likely to be rich enough to cause an interesting dynamical evolution. We will see that this is indeed the case. But before proceeding with this investigation let us interject two remarks (somewhat related to each other).

**Remark 1.** This system (7) is still invariant under translations of the independent variable  $t$  (indeed, it is again *autonomous*) but, in contrast to (1), it is no longer invariant under translations of the dependent variables  $z_n(t)$  nor under a simple rescaling of the independent variable  $t$  and of the dependent variables  $z_n(t)$ .

**Remark 2.** The *general solution* of the equations of motion (7) has the structure

$$z_n(t) = z_{\text{CM}}(t) + \check{z}_n(t), \tag{8a}$$

where the *three* functions  $\check{z}_n(t)$  satisfy themselves the same equations of motion (7) as well as the additional restriction

$$\check{z}_1(t) + \check{z}_2(t) + \check{z}_3(t) = 0 \quad (8b)$$

which is clearly compatible with these equations of motion, and correspondingly  $z_{\text{CM}}(t)$  is the centre of mass of the system (7),

$$z_{\text{CM}}(t) = \frac{z_1(t) + z_2(t) + z_3(t)}{3}, \quad (9a)$$

and it evolves according to the simple formula

$$z_{\text{CM}}(t) = z_{\text{CM}}(0) \exp(-i\omega t) = Z \exp(-i\omega t). \quad (9b)$$

In section 3 (and appendix A) we determine the equilibrium configurations of our *physical* model, namely the values  $z_n^{(\text{eq})}$  of the *three* particle coordinates  $z_n$  such that

$$z_n = z_n^{(\text{eq})}, \quad \dot{z}_n = 0 \quad (10)$$

satisfy the equations of motion (7), and we ascertain the behaviour of our system in the neighbourhood of these configurations. In the second part of section 3, and then almost always in the remainder of this paper (and throughout the rest of this section) we restrict for simplicity consideration to the semisymmetrical case, see (2). A main finding in section 3 (and appendix A) is that in the semisymmetrical case our model (7) possesses generally *two* equilibrium configurations  $z_n^{(\text{eq})}$ . We moreover determine the *three* exponents  $\gamma^{(m)}$  characterizing the small oscillations of our system in the neighbourhood of each of these *two* configurations, defined according to the standard formulae (see section 3)

$$z_n(t) = z_n^{(\text{eq})} + \varepsilon w_n(t), \quad (11a)$$

$$w_n^{(m)}(t) = \exp(-i\gamma^{(m)}\omega t)v_n^{(m)}, \quad (11b)$$

where of course  $\varepsilon$  is an *infinitesimally small* parameter and the quantities  $v_n^{(m)}$  are time-independent. We find that the first *two* of these *three* exponents take in *both* cases the simple values

$$\gamma^{(1)} = 1, \quad \gamma^{(2)} = 2, \quad (12)$$

the first of these corresponding of course to the centre-of-mass motion, see (9b). As for the *third* exponent  $\gamma^{(3)}$ , we find for one equilibrium configuration

$$\gamma^{(3)} = \frac{f + 8g}{f + 2g} = \frac{1}{\mu}, \quad (13a)$$

and for the other

$$\gamma^{(3)} = \frac{f + 8g}{3g} = \frac{2}{1 - \mu}. \quad (13b)$$

Here we have introduced the constant  $\mu$ ,

$$\mu = \frac{f + 2g}{f + 8g} \quad (14)$$

whose value, as we shall see, plays an important role in determining the dynamical evolution of our model: in particular, this evolution does largely depend on whether or not  $\mu$  is a *real rational* number, and if it is *rational*,

$$\mu = \frac{P}{q} \quad (15)$$

with  $p$  and  $q$  coprime integers (and  $q$  positive,  $q > 0$ ), on whether the two natural numbers  $|p|$  and  $q$  are large or small. A hint of this is already apparent from the results we just reported: while the solutions  $w_n^{(1)}(t)$  and  $w_n^{(2)}(t)$ , see (11), are both *periodic* with period  $2T$  (see (5c); in fact  $w_n^{(2)}(t)$  is periodic with period  $T$ ), the solution  $w_n^{(3)}(t)$ , see (11), is periodic with the period  $\tilde{T}$ ,

$$\tilde{T} = \frac{2T}{\gamma^{(3)}}, \tag{16}$$

which is clearly *congruent* to  $T$  only if  $\mu$  is *rational*, see (15) and (13)—implying then that the small oscillations around the equilibrium configurations are always *completely periodic* with a period which is a *finite integer* multiple of  $T$ .

In section 3 we also introduce the special class of (*exact* and *completely explicit*) ‘similarity’ solutions of our equations of motion, (7), and analyse their *stability*, namely the solutions of our system in the immediate neighbourhood of these similarity solutions.

### 2.3. Conserved quantities

It is important to note at this point that the auxiliary model (1) possesses conserved quantities, which will be used in section 5 to obtain its general solution by quadratures. First, due to the translational invariance it is obvious that the quantity

$$Z = \frac{1}{3} \sum_{n=1}^3 \zeta_n \tag{17}$$

does not depend on  $\tau$ . In addition, the analysis of section 5 shows that in the semisymmetrical case there exists an extra conserved quantity given by

$$\tilde{K} = (2\zeta_3 - \zeta_1 - \zeta_2)^{-2} \left[ 1 - \frac{(\zeta_1 - \zeta_2)^2 + (\zeta_2 - \zeta_3)^2 + (\zeta_3 - \zeta_1)^2}{2\mu(2\zeta_3 - \zeta_1 - \zeta_2)^2} \right]^{\mu-1}. \tag{18}$$

Here the constant  $\mu$  is defined in terms of the coupling constants  $g$  and  $f$ , see (2), by (14). We already mentioned that the value of this parameter (in particular, whether or not  $\mu$  is a *rational* number) plays an important role in determining the dynamical evolution of our model. A hint of this is now provided by the appearance of this number  $\mu$  as an exponent on the right-hand side of (18), since this exponent characterizes the multivaluedness of the dependence of the constant  $\tilde{K}$  on the coordinates  $\zeta_n$ .

## 3. Equilibrium configurations, small oscillations and similarity solutions of the physical model

In this section we discuss, first, the equilibrium configurations of our *physical* model, (7), and its behaviour near equilibrium, and second, a special, explicit ‘similarity’ solution of our model and its stability.

The *equilibrium configurations* of our *physical* model (7),

$$z_n(t) = z_n^{(\text{eq})}, \quad \dot{z}_n(t) = 0, \tag{19}$$

(see (10)) are clearly characterized by the algebraic equations

$$i\omega z_n^{(\text{eq})} = \frac{g_{n+1}}{z_n^{(\text{eq})} - z_{n+2}^{(\text{eq})}} + \frac{g_{n+2}}{z_n^{(\text{eq})} - z_{n+1}^{(\text{eq})}}. \tag{20}$$

These algebraic equations entail

$$z_1^{(\text{eq})} + z_2^{(\text{eq})} + z_3^{(\text{eq})} = 0. \tag{21}$$



It is now convenient to set

$$z_n^{(\text{eq})} = (2i\omega)^{-1/2} \alpha_n, \quad (22)$$

so that the equilibrium equations (20) read as follows:

$$\frac{\alpha_n}{2} = \frac{g_{n+1}}{\alpha_n - \alpha_{n+2}} + \frac{g_{n+2}}{\alpha_n - \alpha_{n+1}}. \quad (23)$$

These algebraic equations can be conveniently (see below) rewritten as follows:

$$\alpha_n = \beta_{n+1} (\alpha_n - \alpha_{n+2}) + \beta_{n+2} (\alpha_n - \alpha_{n+1}), \quad (24a)$$

via the position

$$\beta_n = \frac{2g_n}{(\alpha_{n-1} - \alpha_{n+1})^2}. \quad (24b)$$

We now note that, in order that the three equations (24a) (which are linear in the three unknowns  $\alpha_n$ , although only apparently so, see (24b)) have a nonvanishing solution, the quantities  $\beta_n$  must cause the following determinant to vanish:

$$\begin{vmatrix} \beta_2 + \beta_3 - 1 & -\beta_3 & -\beta_2 \\ -\beta_3 & \beta_3 + \beta_1 - 1 & -\beta_1 \\ -\beta_2 & -\beta_1 & \beta_1 + \beta_2 - 1 \end{vmatrix} = 0. \quad (25)$$

To analyse the *small oscillations* of our system (7) around its equilibrium configurations we now set

$$z_n(t) = z_n^{(\text{eq})} + \varepsilon w_n(t) \quad (26a)$$

(see (11a)) and we then get (linearizing by treating  $\varepsilon$  as an infinitesimally small parameter)

$$\dot{w}_n + i\omega w_n + i\omega\beta_{n+1}(w_n - w_{n+2}) + \beta_{n+2}(w_n - w_{n+1}) = 0. \quad (26b)$$

Therefore the three exponents  $\gamma^{(m)}$  characterizing the small oscillations around equilibrium via the formula

$$w_n^{(m)}(t) = \exp(-i\gamma^{(m)}\omega t)v_n^{(m)}, \quad (27)$$

providing *three* independent solutions of the system of *linear* ODEs (26b), are the *three* eigenvalues of the symmetrical matrix

$$\mathbf{B} = \begin{pmatrix} \beta_2 + \beta_3 + 1 & -\beta_3 & -\beta_2 \\ -\beta_3 & \beta_3 + \beta_1 + 1 & -\beta_1 \\ -\beta_2 & -\beta_1 & \beta_1 + \beta_2 + 1 \end{pmatrix}, \quad (28)$$

and the *three* 3-vectors  $\vec{v}^{(m)} \equiv (v_1^{(m)}, v_2^{(m)}, v_3^{(m)})$  are the corresponding eigenvectors,

$$\sum_{\ell=1}^3 B_{n\ell} v_\ell^{(m)} = \gamma^{(m)} v_n^{(m)}. \quad (29)$$

Hence the *three* exponents  $\gamma^{(m)}$  are the *three* roots of the ‘secular equation’

$$\begin{vmatrix} \beta_2 + \beta_3 + 1 - \gamma & -\beta_3 & -\beta_2 \\ -\beta_3 & \beta_3 + \beta_1 + 1 - \gamma & -\beta_1 \\ -\beta_2 & -\beta_1 & \beta_1 + \beta_2 + 1 - \gamma \end{vmatrix} = 0. \quad (30)$$

Clearly these *three* roots are given by the following formulae:

$$\gamma^{(1)} = 1, \quad \gamma^{(2)} = 2, \quad \gamma^{(3)} = 2(\beta_1 + \beta_2 + \beta_3). \quad (31)$$

Indeed the determinant (30) vanishes for  $\gamma = \gamma^{(1)} = 1$  (when each line sums to zero) and for  $\gamma = \gamma^{(2)} = 2$  (see (25)), and the third solution,

$$\gamma^{(3)} = 2(\beta_1 + \beta_2 + \beta_3), \quad (32)$$

is then implied by the trace condition

$$\text{trace}[\mathbf{B}] = 3 + 2(\beta_1 + \beta_2 + \beta_3) = \gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)}. \quad (33)$$

The first of these three solutions,  $\gamma^{(1)} = 1$ , corresponds to the centre-of-mass motion (it clearly entails  $v_n^{(1)} = v^{(1)}$ , see (27) and (28)).

In the semisymmetrical case (2) equations (23) (or equivalently (24)) characterizing, via (22), the equilibrium configurations can be solved explicitly (see appendix A). One finds that there are *two* distinct equilibrium configurations (in fact four, if one takes account of the trivial possibility to exchange the roles of the two ‘equal’ particles with labels 1 and 2), the first of which reads simply

$$z_3^{(\text{eq})} = 0, \quad z_1^{(\text{eq})} = -z_2^{(\text{eq})} = z^{(\text{eq})}, \quad (z^{(\text{eq})})^2 = \frac{f + 2g}{2i\omega}, \quad (34)$$

while the second has a slightly more complicated expression (see appendix A). Note however that, *in both cases*, there holds the relation

$$(z_1^{(\text{eq})} - z_2^{(\text{eq})})^2 + (z_2^{(\text{eq})} - z_3^{(\text{eq})})^2 + (z_3^{(\text{eq})} - z_1^{(\text{eq})})^2 = \frac{3(f + 2g)}{i\omega}. \quad (35)$$

Moreover, in both cases the corresponding values for the eigenvalue  $\gamma^{(3)}$ , see (32), are easily evaluated. The first solution yields (see (13a))

$$\gamma^{(3)} = \frac{f + 8g}{f + 2g} = \frac{1}{\mu} = \frac{q}{p}, \quad (36a)$$

where, for future reference, we expressed  $\gamma^{(3)}$  not only in terms of the parameter  $\mu$ , see (14), but as well in terms of its rational expression (15) (whenever applicable), while the second solution likewise yields

$$\gamma^{(3)} = \frac{f + 8g}{3g} = \frac{2}{1 - \mu} = \frac{2q}{q - p}. \quad (36b)$$

Note that this implies that in the ‘integrable’ case (3) both these formulae, (36a) and (36b), yield  $\gamma^{(3)} = 3$ ; but it is easily seen that in this case only the first equilibrium configuration (34) actually exists. So in the ‘integrable’ case the oscillations around the (only) equilibrium configuration (34) are the linear superposition of three periodic motions (see (27)) with respective periods  $2T$ ,  $T$  and  $\frac{2T}{3}$  (see (5c)). Also in the ‘two-body’ case (4) the second equilibrium configuration does not exist, while the first formula, (36a), yields  $\gamma^{(3)} = 1$ , so in this case the small oscillations around the (only) equilibrium configuration (34) are the linear superposition of two periodic motions, with periods  $2T$  and  $T$  (see (5c); consistently with the explicit solution, easily obtainable from (4c) via (5)).

As can be easily verified, the equilibrium configurations (19) with (20) are merely the special case corresponding to  $z_{\text{CM}}(0) = 0$ ,  $c = 0$  of the following two-parameter family of (*exact*) ‘similarity’ solutions of our equations of motion (7):

$$z_n(t) = z_{\text{CM}}(t) + \tilde{z}_n(t; c), \quad (37a)$$

$$\tilde{z}_n(t; c) \equiv z_n^{(\text{eq})} [1 + c \exp(-2i\omega t)]^{1/2}, \quad (37b)$$

with the centre-of-mass coordinate  $z_{\text{CM}}(t)$  evolving according to (9b). The two *arbitrary* (*complex*) constants featured by this solution are of course  $z_{\text{CM}}(0) = Z$  (see (9b)) and  $c$ , while the constants  $z_n^{(\text{eq})}$ ’s are defined as in the preceding section, see (20).

Clearly these (*exact*) solutions correspond, via the trick (5), relation (22) (which is clearly consistent with (23) and (20)) and the simple relation

$$\tau_b = \frac{c-1}{2i\omega}, \tag{38}$$

to the two-parameter family

$$\zeta_n(\tau) = Z + \alpha_n(\tau - \tau_b)^{1/2}, \tag{39}$$

of (*exact*) solutions of (1).

Let us now discuss the stability of this solution (37b). To this end we set

$$z_n(t) = \tilde{z}_n(t; c) + \varepsilon \tilde{w}_n(t), \tag{40a}$$

and we insert this *ansatz* in our equations of motion (7), linearizing them by treating  $\varepsilon$  as an infinitesimally small parameter. We thus get

$$\dot{\tilde{w}}_n + i\omega \tilde{w}_n + \frac{i\omega[\beta_{n+1}(\tilde{w}_n - \tilde{w}_{n+2}) + \beta_{n+2}(\tilde{w}_n - \tilde{w}_{n+1})]}{1 + c \exp(-2i\omega t)} = 0, \tag{40b}$$

having used the definition (24b). Clearly the solution of this system of ODEs reads

$$\tilde{w}_n(t) = \exp(-i\omega t) \chi_n(\vartheta), \tag{41a}$$

with

$$\vartheta \equiv \vartheta(t) = t - (2i\omega)^{-1} \log \left[ \frac{1 + c \exp(-2i\omega t)}{1 + c} \right] \tag{41b}$$

and the functions  $\chi_n(\vartheta)$  solutions of the autonomous linear system of *first-order* ODEs

$$\chi'_n + i\omega[\beta_{n+1}(\chi_n - \chi_{n+2}) + \beta_{n+2}(\chi_n - \chi_{n+1})] = 0, \tag{41c}$$

where the primes denote of course differentiation with respect to  $\vartheta$ . Hence (see (26b)) the *three* independent solutions of this linear system are

$$\chi_n^{(m)}(\vartheta) = \exp(i\omega\vartheta) w_n^{(m)}(\vartheta), \tag{41d}$$

with the functions  $w_n^{(m)}$  defined by (27) (of course with  $t$  replaced by  $\vartheta$ ), yielding via (27) and (41a) with (41b) the following two equivalent expressions for the *three* independent solutions of the linear system (40b):

$$\tilde{w}_n^{(m)}(t) = \left[ \frac{1 + c \exp(-2i\omega t)}{1 + c} \right]^{(\gamma^{(m)}-1)/2} \exp(i\gamma^{(m)}\omega t) \tilde{v}_n^{(m)}, \tag{42a}$$

$$\tilde{w}_n^{(m)}(t) = \left[ \frac{\exp(2i\omega t) + c}{1 + c} \right]^{(\gamma^{(m)}-1)/2} \exp(i\omega t) \tilde{v}_n^{(m)}. \tag{42b}$$

Here the *three* exponents  $\gamma^{(m)}$  are defined as above, see (31), and likewise the ‘eigenvectors’  $\tilde{v}_n^{(m)}$  coincide with those defined above up to (*arbitrary*) normalization constants  $c^{(m)}$ ,

$$\tilde{v}_n^{(m)} = c^{(m)} v_n^{(m)}. \tag{42c}$$

Note the equivalence of the two expressions (42a) and (42b) (the motivation for writing these two versions of the same formula will be immediately clear).

For  $m = 1, 2, 3$  these solutions, see (42b), are periodic functions of the (*real*) time  $t$  with period  $2T$  if  $|c| > 1$ . If instead  $|c| < 1$ , the solutions (see (42a)) with  $m = 1$  respectively  $m = 2$  are periodic with periods  $2T$  respectively  $T$  (see (31)). The solution with  $m = 3$  is *periodic* if  $\gamma^{(3)}$  is *real*, but with the period  $\frac{2T}{|\gamma^{(3)}|}$  which is not congruent to  $T$  if  $\gamma^{(3)}$  is *irrational*; it grows exponentially with increasing time if  $\text{Im}[\gamma^{(3)}] < 0$ , implying instability

of the solution (37) in this case, and it instead decays exponentially if  $\text{Im}[\gamma^{(3)}] > 0$ , implying a limit cycle behaviour in configuration space, namely asymptotic approach to a solution *completely periodic* with period  $T$  or  $2T$  depending whether the centre of mass of the system is fixed at the origin or itself moving with period  $2T$ ; but note that in this paper we restrict our attention to the case with *real* coupling constants.

#### 4. Analytic structure of the solutions of the *auxiliary* model

In this section we discuss the properties of analyticity as functions of the complex variable  $\tau$  of the solutions  $\zeta_n(\tau)$  of the *auxiliary* model (1) (with arbitrary values of the three coupling constants  $g_n$ , i.e. *not* restricted by the semisymmetrical condition (2): except when this is explicitly specified, see below). In particular we show first of all that, for appropriate initial data characterized by *sufficiently large* values of the moduli of *all three* interparticle distances, namely by the condition (see (6)) that the quantity

$$\zeta_{\min} = \min_{n,m=1,2,3;n \neq m} |\zeta_n(0) - \zeta_m(0)| \tag{43}$$

be *adequately large*, the solutions  $\zeta_n(\tau)$  are *holomorphic* in a disk  $D_0$  of (*arbitrarily large*) radius  $d_0$  centred at the origin,  $\tau = 0$ , of the complex  $\tau$ -plane (of course the ‘adequately large’ value of the quantity  $\zeta_{\min}$  depends on  $d_0$ , and on the magnitude of the *three* coupling constants  $g_n$ ; see (51) below). We moreover discuss via a local analysis *a la Painlevé* the nature of the singularities of the solutions  $\zeta_n(\tau)$  of the *auxiliary* model (1) as functions of the *complex* variable  $\tau$  and we thereby justify the assertions made in this respect in section 2.

To prove the first point, set

$$\sigma_n(\tau) = \zeta_n(\tau) - \zeta_n(0), \tag{44a}$$

so that these quantities  $\sigma_n(\tau)$  vanish initially,

$$\sigma_n(0) = 0, \tag{44b}$$

and, as a consequence of (1), satisfy the equations of motion

$$\sigma_n'(\tau) = \frac{g_{n+1}}{\zeta_n(0) - \zeta_{n+2}(0) + \sigma_n(\tau) - \sigma_{n+2}(\tau)} + \frac{g_{n+2}}{\zeta_n(0) - \zeta_{n+1}(0) + \sigma_n(\tau) - \sigma_{n+1}(\tau)}. \tag{44c}$$

A standard theorem (see, for instance, [18]) guarantees then that these quantities  $\sigma_n(\tau)$ —hence as well the functions  $\zeta_n(\tau)$ , see (44a)—are *holomorphic* in  $\tau$  (at least) in a disk  $D_0$  centred at the origin  $\tau = 0$  in the *complex*  $\tau$ -plane, the radius  $d_0$  of which is bounded *below* by the inequality

$$d_0 > \frac{b}{4M} \tag{45}$$

(this formula coincides with the last equation of section 13.21 of [18], with the assignments  $m = 3$  and  $a = \infty$ , the first of which is justified by the fact that the system (44c) features three coupled equations, the second of which is justified by the *autonomous* character of the equations of motion (44c)). The two *positive* quantities  $b$  and  $M$  on the right-hand side of this inequality are defined as follows. The quantity  $b$  is required to guarantee that the right-hand sides of the equations of motion (44c) be *holomorphic* (as functions of the dependent variables  $\sigma_n$ ) provided these quantities satisfy the three inequalities

$$|\sigma_n| \leq b, \tag{46}$$

clearly in our case a sufficient condition to guarantee this is provided by the single restriction

$$b < \frac{\zeta_{\min}}{2}, \tag{47}$$

with  $\zeta_{\min}$  defined by (43). The second quantity on the right-hand side of (45),  $M \equiv M(b)$ , is the *upper* bound of the right-hand sides of (44c) when the quantities  $\sigma_n$  satisfy the inequality (46); but of course the inequality (45) holds *a fortiori* if we overestimate  $M$ , as we shall presently do. Indeed clearly the equations of motion (44c) with (46) and (47) entail

$$M < \frac{4G}{\zeta_{\min} - 2b}, \tag{48}$$

with

$$G = \max_{n=1,2,3} |g_n|. \tag{49}$$

Insertion of (48) into (45) yields

$$d_0 > \frac{b(\zeta_{\min} - 2b)}{16G}, \tag{50}$$

hence, setting  $b = \frac{\zeta_{\min}}{4}$  (to maximize the right-hand side; note the consistency of this assignment with (47)),

$$d_0 > \frac{\zeta_{\min}^2}{128G}, \tag{51}$$

confirming the assertion made above (that  $d_0$  can be made *arbitrarily large* by choosing  $\zeta_{\min}$  *adequately large*).

Next, let us show, via a local analysis *à la Painlevé*, that the singularities as functions of the complex variable  $\tau$  of the *general* solutions  $\zeta_n(\tau)$  of our *auxiliary* model (1) associated with a coincidence of *two* of the *three* components  $\zeta_n$  are *square-root branch points* (recall that a singularity at finite  $\tau$  of a solution  $\zeta_n(\tau)$  of the evolution equations (1) may only occur when the right-hand side of these equations diverges). Such a singularity occurs for those values  $\tau_b$  of the independent variable  $\tau$  such that *two* of the *three* functions  $\zeta_n$  coincide, for instance

$$\zeta_1(\tau_b) = \zeta_2(\tau_b) \neq \zeta_3(\tau_b). \tag{52}$$

The *square-root* character of these branch points is evident from the following *ansatz* characterizing the behaviour of the solutions of (1) in the neighbourhood of these singularities:

$$\zeta_s(\tau) = \zeta_b - (-1)^s \alpha(\tau - \tau_b)^{1/2} + v_s(\tau - \tau_b) + \sum_{k=3}^{\infty} \alpha_s^{(k)}(\tau - \tau_b)^{k/2}, \quad s = 1, 2 \tag{53a}$$

$$\zeta_3(\tau) = \zeta_{3b} + v_3(\tau - \tau_b) + \sum_{k=3}^{\infty} \alpha_3^{(k)}(\tau - \tau_b)^{k/2},$$

with

$$\alpha^2 = g_3, \quad v_3 = -\frac{g_1 + g_2}{\zeta_b - \zeta_{3b}}, \quad v_s = \frac{g_s + 5g_{s+1}}{6(\zeta_b - \zeta_{3b})}, \quad s = 1, 2 \text{ mod}(2), \tag{53b}$$

and the constants  $\alpha_n^{(k)}$  determinable (in principle) recursively (for  $k = 3, 4, \dots$ ) by inserting this *ansatz* in (1), so that, to begin with

$$\alpha_3^{(3)} = \frac{2\alpha(g_2 - g_1)}{3(\zeta_b - \zeta_3)^2}, \tag{53c}$$

$$\alpha_s^{(3)} = -(-1)^s \frac{\alpha}{36(\zeta_b - \zeta_3)^2} \left[ 3(g_s - 7g_{s+1}) + \frac{(g_1 - g_2)^2}{g_3} \right], \tag{53d}$$

$s = 1, 2 \text{ mod}(2),$

and so on. The diligent reader will verify the consistency of this procedure, for any assignment of the *three* constants  $\tau_b, \zeta_b$  and  $\zeta_{3b}$ , which remain undetermined except for the obvious restrictions  $\tau_b \neq 0, \zeta_b \neq 0, \zeta_{3b} \neq \zeta_b$ . The fact that (53) contains *three* arbitrary (*complex*) constants—the maximal number of integration constants compatible with the system of *three* first-order ODEs (1)—shows that this *ansatz* is indeed adequate to represent locally, in the neighbourhood of its singularities occurring at  $\tau = \tau_b$ , the *general* solution of (1).

An analogous analysis of the behaviour of the solutions of the system (1) near the values of the independent variable  $\tau$  where a *triple* coincidence of all *three* functions  $\zeta_n$  occurs (corresponding to the excluded assignment  $\zeta_{3b} = \zeta_b$  in the above *ansatz* (53)), indicates, somewhat surprisingly, that such a *triple* coincidence,

$$\zeta_1(\tau_b) = \zeta_2(\tau_b) = \zeta_3(\tau_b) = Z \tag{54}$$

might also occur for the *general* solution of the system (1). This conclusion is reached via a local analysis analogous to that performed above, and is then confirmed (for the semisymmetrical case, see (2)) by the *exact* treatment of section 5. Indeed the natural extension of the above *ansatz* (53) characterizing the behaviour of the solutions of (1) in the neighbourhood of such singularities, corresponding to a *triple* coincidence, see (54), of the *three* functions  $\zeta_n(\tau)$ , reads as follows:

$$\zeta_n(\tau) = Z + \eta_n(\tau - \tau_b)^{(1-\gamma)/2} + \alpha_n(\tau - \tau_b)^{1/2} + o(|\tau - \tau_b|^{1/2}), \tag{55a}$$

provided

$$\text{Re}(\gamma) < 0. \tag{55b}$$

Here the *three* constants  $\alpha_n$  are determined, as can be easily verified, just by the *three* nonlinear algebraic equations (23) that were found in the preceding section while investigating the equilibrium configurations of our *physical* system (7), while the *three* constants  $\eta_n$ , as well as the exponent  $\gamma$ , are required to satisfy the algebraic equations

$$\frac{(\gamma - 1)\eta_n}{2} = \frac{g_{n+1}(\eta_n - \eta_{n+2})}{(\alpha_n - \alpha_{n+2})^2} + \frac{g_{n+2}(\eta_n - \eta_{n+1})}{(\alpha_n - \alpha_{n+1})^2}. \tag{56}$$

These algebraic equations, (23) and (56), can be conveniently rewritten as follows:

$$\alpha_n = \beta_{n+1}(\alpha_n - \alpha_{n+2}) + \beta_{n+2}(\alpha_n - \alpha_{n+1}), \tag{57a}$$

$$(\gamma - 1)\eta_n = \beta_{n+1}(\eta_n - \eta_{n+2}) + \beta_{n+2}(\eta_n - \eta_{n+1}), \tag{57b}$$

via the introduction of the quantities  $\beta_n$ , see (24b). Note that in this manner these two sets of equations, (57a) and (57b), have a quite similar look, which should however not mislead the reader to underestimate their basic difference: the *three* equations (57a) are merely a convenient way to rewrite, via the definition (24b), the *three nonlinear* equations (23), which determine (albeit not uniquely, see appendix A) the *three* constants  $\alpha_n$ ; while equations (57b) are *three linear* equations for the *three* quantities  $\eta_n$ , hence they can determine these *three* unknowns only up to a common multiplicative constant (provided they admit a *nontrivial* solution: see below).

Of course these linear equations (57b) admit the *trivial* solution  $\eta_n = 0$ , and it is easily seen that there indeed is a special (*exact*) solution of the equations of motion (1) having this property, see (39) with the constants  $\alpha_n$  determined by (23) and computed, for the semisymmetrical model, in appendix A. This ‘similarity solution’ (39) of the system (1) has been discussed in the preceding section, but let us emphasize here that it only provides a *two*-parameter ( $Z$  and  $\tau_b$ ) class of solutions of the equations of motion (1), while the *general* solution of this system of *three* first-order ODEs must of course feature *three* arbitrary parameters.

A *general* solution of the evolution equations (1) corresponds instead to the *ansatz* (55a) if the linear equations (57b) for the *three* coefficients  $\eta_n$  admit a *nonvanishing* solution, because in such a case, as mentioned above, a common scaling parameter for these *three* coefficients remains as an additional (*third*) free parameter (besides  $Z$  and  $\tau_b$ ). The condition for this to happen is the vanishing of the determinant of the coefficients of these *three* linear equations, (57b), namely again validity of the determinantal condition (30), a *cubic* equation for the unknown  $\gamma$ , which determines, as discussed in the preceding section, the *three* values (31) of this quantity. But the first two of these values,  $\gamma = \gamma^{(1)} = 1$  and  $\gamma = \gamma^{(2)} = 2$  (see (31)), are *not* consistent with the requirement (55b). The third solution,  $\gamma = \gamma^{(3)} = 2(\beta_1 + \beta_2 + \beta_3)$  (see (31)) might instead be consistent with the requirement (55b), and whenever this happens the *ansatz* (55) indicates that the *general* solution of the system of ODEs (1) does feature a ‘triple coincidence’, see (54), and identifies the character of the corresponding *branch point*.

In the semisymmetrical case (2) the equations characterizing the equilibrium configuration, (23) or equivalently (57), can be solved (see appendix A). One finds that there are *two* distinct solutions of these nonlinear equations (23) (in fact *four*, since each solution has a trivial counterpart obtained by exchanging the role of the two ‘equal’ particles with labels 1 and 2). The *first* solution yields for  $\gamma = \gamma^{(3)}$  the value (36a), which is consistent with the condition (55b) iff

$$\text{Re}(\mu) < 0, \tag{58}$$

and it yields for the branch point exponent, see (55b), the value

$$\frac{1 - \gamma}{2} = \frac{\mu - 1}{2\mu} = \frac{p - q}{2p}, \tag{59}$$

while the *second* solution yields for  $\gamma^{(3)}$  the value (36b), which is consistent with the condition (55b) iff

$$\text{Re}(\mu) > 1, \tag{60}$$

and it yields for the branch point exponent, see (55a), the value

$$\frac{1 - \gamma}{2} = \frac{\mu + 1}{2(\mu - 1)} = \frac{p + q}{2(p - q)}. \tag{61}$$

The last equality in (59) and (61) are of course only valid if  $\mu$  is rational,  $\mu = p/q$ .

Note that these findings imply that the branch point associated with ‘triple coincidences’ is not (only) of *square-root* type, being also characterized, see (55a), by the exponent  $\frac{1-\gamma}{2}$ , the value of which depends on the value of the parameter  $\mu$ , see (59) and (61); however this kind of branch point is *not* present if

$$0 < \text{Re}(\mu) < 1, \tag{62}$$

since in this case neither (58) nor (60) is satisfied.

The results presented in this section are not entirely rigorous, since the local analysis of the singularities we performed above on the basis of appropriate *ansätze* should be complemented by proofs that the relevant expansions converge. Moreover these analyses provide information on the nature of the branch points, but neither on their number nor at their location. But these results are confirmed and complemented below (see section 5) by the analysis of the *exact* general solution of the equations of motion (1). Our motivation for having nevertheless presented here a discussion of the character of the singularities of the solutions of (1) via a local analysis *à la Painlevé* is because an analogous treatment may be applicable to models which are not as explicitly solvable as that treated in this paper (see for instance [11] and [9]).

### 5. General solution by quadratures

In this section we obtain and discuss the *general solutions* of our models, (1) and (7). But since the *general solution* of the *physical* model (7) is easily obtained via the trick (5) from the *general solution* of the *auxiliary* problem (1), we focus to begin with on this model.

A first constant of the motion is provided by the centre-of-mass coordinate

$$Z = \frac{\zeta_1 + \zeta_2 + \zeta_3}{3}, \tag{63a}$$

since the equations of motion (1) clearly entail

$$Z' = 0 \tag{63b}$$

hence

$$Z(\tau) = Z(0). \tag{63c}$$

And clearly the *general solution* of (1) reads

$$\zeta_n(\tau) = Z + \check{\zeta}_n(\tau), \tag{64a}$$

with the set of three functions  $\check{\zeta}_n(\tau)$  providing themselves a solution of (1), independent of the value of  $Z$  and satisfying the (compatible) constraint

$$\check{\zeta}_1(\tau) + \check{\zeta}_2(\tau) + \check{\zeta}_3(\tau) = 0. \tag{64b}$$

It is moreover clear that the equations of motion (1) entail

$$\zeta_1' \zeta_1 + \zeta_2' \zeta_2 + \zeta_3' \zeta_3 = g_1 + g_2 + g_3, \tag{65a}$$

hence there also holds the relation

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 2(g_1 + g_2 + g_3)(\tau - \tau_0). \tag{65b}$$

It is now convenient to set, as in appendix B of [11],

$$\zeta_s = Z - \left(\frac{2}{3}\right)^{1/2} \rho \cos \left[ \theta - (-1)^s \frac{2\pi}{3} \right], \quad s = 1, 2, \tag{66a}$$

$$\zeta_3 = Z - \left(\frac{2}{3}\right)^{1/2} \rho \cos \theta. \tag{66b}$$

Then, summing the squares of these three formulae and using the *identities*

$$\cos(\theta) + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right) = 0, \tag{67}$$

$$\cos^2(\theta) + \cos^2\left(\theta + \frac{2\pi}{3}\right) + \cos^2\left(\theta - \frac{2\pi}{3}\right) = \frac{3}{2}, \tag{68}$$

one easily gets

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 3Z^2 + \rho^2 \tag{69a}$$

or equivalently

$$\rho^2 = \frac{1}{3}[(\zeta_1 - \zeta_2)^2 + (\zeta_2 - \zeta_3)^2 + (\zeta_3 - \zeta_1)^2], \tag{69b}$$

hence, from (65b),

$$\rho^2 = 2(g_1 + g_2 + g_3)(\tau - \tau_0) - 3Z^2 = 2(g_1 + g_2 + g_3)(\tau - \tau_1), \tag{69c}$$



$$\tau_1 = \tau_0 + \frac{3Z^2}{2(g_1 + g_2 + g_3)}, \tag{69d}$$

which also entails

$$\rho' \rho = g_1 + g_2 + g_3. \tag{69e}$$

Here we assume that the sum of the *three* coupling constants  $g_n$  does not vanish,  $g_1 + g_2 + g_3 \neq 0$ . The special case in which this sum does instead vanish is treated in appendix B. The expression of the constant  $\tau_1$  in terms of the initial data is of course (see (69c))

$$\tau_1 = -\frac{\rho^2(0)}{2(g_1 + g_2 + g_3)}, \tag{70a}$$

namely (see (69c))

$$\tau_1 = -\frac{\zeta_1^2(0) + \zeta_2^2(0) + \zeta_3^2(0) - 3Z^2}{2(g_1 + g_2 + g_3)}, \tag{70b}$$

or equivalently (see (69d))

$$\tau_1 = -\frac{(\zeta_1 - \zeta_2)^2 + (\zeta_2 - \zeta_3)^2 + (\zeta_3 - \zeta_1)^2}{6(g_1 + g_2 + g_3)}. \tag{70c}$$

There remains to compute  $\theta(\tau)$ , or rather

$$u(\tau) = \cos \theta(\tau). \tag{71}$$

Inserting the *ansatz* (66) in the equations of motion (1) with  $n = 3$ , one easily gets

$$\rho^2(\cos \theta)'(4 \cos^2 \theta - 1) = (4g_1 + 4g_2 + g_3) \cos \theta - 4(g_1 + g_2 + g_3) \cos^3 \theta + \sqrt{3}(g_1 - g_2) \sin \theta. \tag{72}$$

From now on in this section—for simplicity, and because it is sufficient for our purposes—we restrict attention to the semisymmetrical case (2), so that the last equation becomes simply, via (71),

$$\rho^2 u'(4u^2 - 1) = (f + 8g)u - 4(f + 2g)u^3. \tag{73}$$

The general case *without* the restriction (2) is treated in appendix B.

This ODE can be easily integrated via a quadrature (using (69c)), and this leads to the following formula:

$$[u(\tau)]^{-2\mu} \left[ u^2(\tau) - \frac{1}{4\mu} \right]^{\mu-1} = K(\tau - \tau_1), \tag{74}$$

where the parameter  $\mu$  is defined by (14) and  $K$  is an integration constant. Here we are of course assuming that  $f + 8g \neq 0$  (see (14)); the case when this does *not* happen is treated in appendix B. (Also recall that, as promised above, we shall treat in appendix C the case in which the sum of the three coupling constants  $g_n$  vanishes, namely when  $f + 2g = 0$ , which entails  $\mu = 0$ , see (14)). As for the quantity  $K$  in (74), it is an (*a priori arbitrary*) integration constant. It is a matter of elementary algebra to express this constant in terms of the original dependent variables  $\zeta_n$  (via (74), (69c), (71) and (66)), and one thereby obtains the relation

$$K = 12(f + 2g)\tilde{K} \tag{75}$$

with  $\tilde{K}$  defined by (18). This finding justifies the assertion that  $\tilde{K}$  is a constant of motion, see section 2.3; and of course it determines the value to be assigned to the constant  $K$  in the context of the *initial-value* problem. Likewise the value to be assigned, in the context of the

*initial-value* problem, to the constant  $\tau_1$  appearing on the right-hand side of (74) is given by the formula

$$K \tau_1 = -[u(0)]^{-2\mu} \left[ u^2(0) - \frac{1}{4\mu} \right]^{\mu-1}, \quad (76)$$

where (see (71) and (66b))

$$u(0) = - \left( \frac{3}{2} \right)^{1/2} \frac{\zeta_3(0) - Z}{\rho(0)}, \quad (77a)$$

namely

$$u(0) = - \frac{2\zeta_3(0) - \zeta_1(0) - \zeta_2(0)}{[2\{[\zeta_1(0) - \zeta_2(0)]^2 + [\zeta_2(0) - \zeta_3(0)]^2 + [\zeta_3(0) - \zeta_1(0)]^2\}]^{1/2}}. \quad (77b)$$

Of course in these formulae the initial values  $\zeta_n(0)$  of the coordinates  $\zeta_n(\tau)$  of the *auxiliary* problem (1) can be replaced by the initial values  $z_n(0)$  of the *physical* problem (7), see (6).

Let us emphasize that we have now reduced, via (66) with (69c) and (71), the solution of our problem (1) to the investigation of the function  $u(\tau)$  of the *complex* variable  $\tau$ , defined for  $\tau \neq 0$  as the solution of the (*nondifferential*) equation (74) that evolves by continuity from  $u(0)$  at  $\tau = 0$ .

To proceed with our analysis an additional change of variables is now convenient. We introduce the new (*complex*) independent variable  $\xi$  by setting

$$\xi = \frac{K(\tau - \tau_1)}{4\mu}, \quad (78)$$

and the new (*complex*) dependent variable  $w \equiv w(\xi)$  by setting

$$w(\xi) = 4\mu[u(\tau)]^2. \quad (79)$$

Thereby the expression of the solution (66) of our original problem (1) reads

$$\zeta_s(\tau) = Z - \left( \frac{f + 2g}{3K} \right)^{1/2} \xi^{1/2} \{ -[w(\xi)]^{1/2} + (-)^s [12\mu - 3w(\xi)]^{1/2} \}, \quad (80a)$$

$s = 1, 2,$

$$\zeta_3(\tau) = Z - 2 \left( \frac{f + 2g}{3K} \right)^{1/2} [\xi w(\xi)]^{1/2}, \quad (80b)$$

while the (*nondifferential*) equation that determines the dependence of  $w(\xi)$  on the (*complex*) variable  $\xi$  reads

$$[w(\xi) - 1]^{\mu-1} [w(\xi)]^{-\mu} = \xi. \quad (81)$$

Note that this equation is *independent* of the *initial data*; it only features the constant  $\mu$ , which *only* depends on the coupling constants, see (14).

We conclude that the solution of our *physical* problem (7) as the *real* time variable  $t$  evolves onwards from  $t = 0$  is essentially given, via (80) and (5), by the evolution of the solution  $w(\xi)$  of this (*nondifferential*) equation, (81), as the *complex* variable  $\xi$  travels round and round on the circle  $\Xi$  in the *complex*  $\xi$ -plane defined by the equation (see (78) and (5a))

$$\xi = R \exp(2i\omega t) + \bar{\xi} = R[\exp(2i\omega t) + \eta], \quad (82a)$$

namely on the circle with centre  $\bar{\xi}$  and radius  $|R|$ . The parameters  $R$  and  $\bar{\xi}$  (or  $\eta$ ) depend on the initial data according to the formulae (implied by (78), (5), (75), (18), (70c))

$$R = \frac{3(f + 8g)}{2i\omega[2z_3(0) - z_1(0) - z_2(0)]^2} [1 - \kappa]^{\mu-1}, \tag{82b}$$

$$\bar{\xi} = R\eta, \tag{82c}$$

$$\eta = \frac{i\omega\{[\zeta_1(0) - \zeta_2(0)]^2 + [\zeta_2(0) - \zeta_3(0)]^2 + [\zeta_3(0) - \zeta_1(0)]^2\}}{3(f + 2g)} - 1, \tag{82d}$$

$$\kappa = \frac{2\mu[2\zeta_3(0) - \zeta_1(0) - \zeta_2(0)]^2}{[\zeta_1(0) - \zeta_2(0)]^2 + [\zeta_2(0) - \zeta_3(0)]^2 + [\zeta_3(0) - \zeta_1(0)]^2}. \tag{82e}$$

Of course in these formulae the initial values  $\zeta_n(0)$  of the coordinates  $\zeta_n(\tau)$  of the *auxiliary* problem (1) can be replaced by the initial values  $z_n(0)$  of the coordinates  $z_n(t)$  of the *physical* problem (7), see (6).

Let us emphasize that, as the complex variable  $\xi$  travels on the circle  $\Xi$ —taking the time  $T$  to make each round, see (82a) and (5c)—the dependent variable  $w$  ( $\xi$ ) travels on the Riemann surface determined by its dependence on the *complex* variable  $\xi$ , as entailed by equation (81) that relates  $w$  ( $\xi$ ) to its argument  $\xi$ —starting at  $t = 0$  from  $\xi = \xi_0$ ,

$$\xi_0 = \bar{\xi} + R = (\eta + 1)R, \tag{83a}$$

$$\xi_0 = \frac{i\omega R\{[\zeta_1(0) - \zeta_2(0)]^2 + [\zeta_2(0) - \zeta_3(0)]^2 + [\zeta_3(0) - \zeta_1(0)]^2\}}{3(f + 2g)} \tag{83b}$$

(see (82)) and correspondingly from  $w(\xi_0) = w_0$ ,

$$w_0 = \frac{1}{\kappa} = \frac{[\zeta_1(0) - \zeta_2(0)]^2 + [\zeta_2(0) - \zeta_3(0)]^2 + [\zeta_3(0) - \zeta_1(0)]^2}{2\mu[2\zeta_3(0) - \zeta_1(0) - \zeta_2(0)]^2} \tag{84}$$

(see (82e)).

Let us therefore now discuss the structure of this Riemann surface, namely the analytic properties of the function  $w(\xi)$  defined by (81). There are two types of singularities, the ‘fixed’ ones occurring at values of the independent variable  $\xi$ , and correspondingly of the dependent variable  $w$ , that can be read directly from the structure of equation (81) under investigation, and the ‘movable’ ones (this name being given to underline their difference from the *fixed* ones) occurring at values of the independent and dependent variables,  $\xi$  and  $w$ , that cannot be directly read from the structure of equation (81) under investigation (they ‘move’ as the initial data are modified).

### 5.1. Movable singularities

To investigate their nature it is convenient to differentiate (81), obtaining thereby (using again (81))

$$\xi w' = -\frac{w(w - 1)}{w - \mu}, \tag{85}$$

where the prime indicates of course differentiation with respect to  $\xi$ . (Note that this ODE is implied by the nondifferential equation (81), while its solution reproduces the nondifferential equation (81) up to multiplication of its right-hand side by an arbitrary constant.) The position of the singularities,  $\xi_b$ , and the corresponding values of the dependent variable,  $w_b \equiv w(\xi_b)$ , are then characterized by the vanishing of the denominator on the right-hand side of this formula, yielding the relation

$$w_b = \mu, \tag{86}$$

which, combined with (81) (at  $\xi = \xi_b$ ) is easily seen to yield

$$\xi_b = \xi_b^{(k)} = r \exp(2\pi i \mu k), \quad k = 1, 2, 3, \dots, \tag{87a}$$

$$\xi_b = \xi_b^{(k)} = r \exp\left[i \frac{2\pi p k}{q}\right], \quad k = 1, 2, \dots, q, \tag{87b}$$

$$r = (\mu - 1)^{-1} \left(\frac{\mu - 1}{\mu}\right)^\mu. \tag{87c}$$

In the last, (87c), of these formulae it is understood that the principal determination is to be taken of the  $\mu$ th power appearing on the right-hand side. The first of these formulae, (87a), shows clearly that the number of these branch points is *infinite* if the parameter  $\mu$  is *irrational*, and that they then sit densely on the circle  $B$  in the complex  $\xi$ -plane centred at the origin and having radius  $r$ , see (87c). Note that this entails that the *generic* point on the circle  $B$  is *not* a branch point (just as a *generic real* number is *not rational*), but every generic point on the circle  $B$  has some branch point (in fact, an *infinity* of branch points!) *arbitrarily* close to it (just as every generic *real* number has an *infinity* of *rational* numbers *arbitrarily* close to it). As for the second of this formulae, (87b), it is instead appropriate to the case in which the parameter  $\mu$  is *rational*, see (15), in which case the branch points sit again on the circle  $B$  in the complex  $\xi$ -plane, but there are only a *finite* number,  $q$ , of them.

These singularities are all *square-root* branch points, as implied by the following standard proof. Set, for  $\xi \approx \xi_b$ ,

$$w(\xi) = \mu + a(\xi - \xi_b)^\beta + o(|\xi - \xi_b|^{\text{Re}(\beta)}), \tag{88a}$$

with the assumption (immediately verified, see below) that

$$0 < \text{Re}(\beta) < 1. \tag{88b}$$

It is then immediately seen that the insertion of this *ansatz* in (85) (is consistent and) yields

$$\beta = \frac{1}{2}, \quad a^2 = \frac{2(1 - \mu)}{\xi_b} = -2 \left(\frac{\mu}{\mu - 1}\right)^\mu. \tag{88c}$$

Note that these results confirm the treatment of section 4: the *square-root* branch points of  $w(\xi)$  identified here, see (86), are easily seen to correspond, via (80), to the pair coincidence  $\zeta_1(\tau_b) = \zeta_3(\tau_b)$  or  $\zeta_2(\tau_b) = \zeta_3(\tau_b)$ , while there is an additional class of *square-root* branch points which only affect  $\zeta_1(\tau)$  and  $\zeta_2(\tau)$ , but neither  $\zeta_3(\tau)$  nor  $w(\xi)$ , and occur at

$$w = 4\mu \tag{89a}$$

due to the vanishing of the second *square-root* term inside the curly bracket on the right-hand side of (80a), and correspond therefore to the coincidence  $\zeta_1(\tau_b) = \zeta_2(\tau_b)$ . The corresponding values of  $\xi$  (as implied by (89a) with (81)) are

$$\xi = \frac{(4\mu - 1)^{\mu-1}}{(4\mu)^\mu} = \frac{1}{4\mu} \left(1 - \frac{1}{4\mu}\right)^{\mu-1} \tag{89b}$$

(we use the plural to refer to these values because of the multivaluedness of the function on the right-hand side of this formula).

### 5.2. Fixed singularities

Next, let us consider the ‘fixed’ singularities, which clearly can only occur at  $\xi = \infty$  and at  $\xi = 0$ , with corresponding values of  $w$ .

Let us investigate first the nature of the singularities at  $\xi = \infty$ . Two behaviours of  $w(\xi)$  are then possible for  $\xi \approx \infty$ , depending on the value of (the real part of)  $\mu$ . The first is

characterized by the *ansatz*

$$w(\xi) = a\xi^\beta + o(|\xi|^{\operatorname{Re}(\beta)}), \quad \operatorname{Re}(\beta) < 0, \quad (90a)$$

and its insertion in (81) yields

$$\beta = -\frac{1}{\mu}, \quad a^\mu = -\exp(i\pi\mu), \quad (90b)$$

which is consistent with (90a) iff

$$\operatorname{Re}(\mu) > 0. \quad (90c)$$

The second is characterized by the *ansatz*

$$w(\xi) = 1 + a\xi^\beta + o(|\xi|^{\operatorname{Re}(\beta)}), \quad \operatorname{Re}(\beta) < 0, \quad (91a)$$

and its insertion in (81) yields

$$\beta = \frac{1}{\mu - 1}, \quad a^{\mu-1} = 1, \quad (91b)$$

which is consistent with (91a) iff

$$\operatorname{Re}(\mu) < 1. \quad (91c)$$

We therefore conclude that there are three possibilities: if  $\operatorname{Re}(\mu) > 1$ , only the first *ansatz*, (90), is applicable, and it characterizes the nature of the branch point of  $w(\xi)$  at  $\xi = \infty$ ; if  $\operatorname{Re}(\mu) < 0$ , only the second *ansatz*, (91), is applicable, and it characterizes the nature of the branch point of  $w(\xi)$  at  $\xi = \infty$ ; while if  $0 < \operatorname{Re}(\mu) < 1$ , both *ansätze*, (90) and (91), are applicable, so both types of branch points occur at  $\xi = \infty$ .

Next, let us investigate the nature of the singularity at  $\xi = 0$ . It is then easily seen, by an analogous treatment, that two behaviours are possible, as displayed by the following *ansätze*: either

$$w(\xi) = a\xi^\beta + o(|\xi|^{\operatorname{Re}(\beta)}), \quad \operatorname{Re}(\beta) > 0, \quad (92a)$$

$$\beta = -\frac{1}{\mu}, \quad a^\mu = -\exp(i\pi\mu), \quad (92b)$$

which is applicable iff

$$\operatorname{Re}(\mu) < 0; \quad (92c)$$

or

$$w(\xi) = 1 + a\xi^\beta + o(|\xi|^{\operatorname{Re}(\beta)}), \quad \operatorname{Re}(\beta) > 0, \quad (93a)$$

$$\beta = \frac{1}{\mu - 1}, \quad a^{\mu-1} = 1, \quad (93b)$$

which is applicable iff

$$\operatorname{Re}(\mu) > 0. \quad (93c)$$

This analysis shows that the function  $w(\xi)$  features a *branch point* at  $\xi = 0$  the nature of which is characterized by the relevant exponent  $\beta$ , see (92b) or (93b), whichever is applicable (see (92c) and (93c)). But let us emphasize that there is *no* branch point *at all* at  $\xi = 0$  if neither one of the two inequalities (92c) and (93c) holds, namely if  $0 < \operatorname{Re}(\mu) < 1$ .

### 5.3. Explicitly solvable cases

Let us end section 5 by noting that equation (81) for certain rational values of  $\mu$  reduces to such a low degree polynomial equation that it can be solved *explicitly*. In particular, the polynomial equation is of second degree if  $\mu = -1, 1/2$  or  $2$ ; it is of third degree if  $\mu = -2, -1/2, 1/3, 2/3, 3/2$ , or  $3$ ; while it is of fourth degree if  $\mu = -3, -1/3, 1/4, 1/2, 3/4, 4/3$  or  $4$ .

The diligent reader might wish to use the corresponding *explicit* solutions formulae for these cases to verify the validity of the previous discussion.

## 6. The physical model

The solution (80) can also be written, via (75), (18) and (5), directly for the particle coordinates  $z_n(t)$ , to read as follows:

$$z_s(t) = Z e^{i\omega t} - \frac{2z_3(0) - z_1(0) - z_2(0)}{6\sqrt{\mu}} [\eta \exp(-2i\omega t) + 1]^{1/2} \cdot \\ \times (-[\check{w}(t)]^{1/2} + (-)^s [12\mu - 3\check{w}(t)]^{1/2}), \quad s = 1, 2, \quad (94a)$$

$$z_3(t) = Z e^{i\omega t} - \frac{2z_3(0) - z_1(0) - z_2(0)}{3\sqrt{\mu}} [\eta \exp(-2i\omega t) + 1]^{1/2} [\check{w}(t)]^{1/2}, \quad (94b)$$

where the constant  $\eta$  is given in terms of the initial data by (82d) and we set

$$\check{w}(t) \equiv w[\xi(t)], \quad (95)$$

so that this dependent variable is now the solution of the nondifferential equation (see (81))

$$[\check{w}(t) - 1]^{\mu-1} [\check{w}(t)]^{-\mu} = R \exp(2i\omega t) + \bar{\xi} = R[\exp(2i\omega t) + \eta], \quad (96)$$

where the constants  $R, \bar{\xi}$  and  $\eta$  are defined in terms of the initial data, see (82) (and recall that the initial data  $\zeta_n(0)$  can be replaced by the initial data  $z_n(0)$ , see (6)). The dependent variable  $\check{w}(t)$  is of course the solution of this equation, (96), identified by continuity, as the time  $t$  unfolds from  $t = 0$ , from the initial datum  $\check{w}(0) = w_0$  assigned at  $t = 0$ , see (84): this specification is necessary, since generally the nondifferential equation (81) has more than a single solution, in fact possibly even an infinity of solutions.

A discussion of the behaviour of this solution of the initial-value problem of our model (7) with (2) clearly hinges on ascertaining how the solution  $\check{w}(t)$  of (96) evolves in time. Equation (96) corresponds of course to the combination of (81) and (82). Hence one must firstly elucidate the structure of the Riemann surface defined by the dependence of  $w(\xi)$  on the complex variables  $\xi$  as determined by the nondifferential equations (81), and then understand the consequences of a travel on this Riemann surface when the complex variable  $\xi$  evolves according to (82), namely it travels round and round on the circle of centre  $\bar{\xi}$  and radius  $|R|$  in the complex  $\xi$ -plane.

The first task is simple, its foundation being provided by the analysis provided in sections 5.1 and 5.2.

The second task is much more demanding, inasmuch as it hinges on the detailed manner the sheets of the Riemann surface are connected via the cuts associated with the branch points discussed in sections 5.1 and 5.2. The main results of this analysis have already been reported (without proofs) in [1]; their derivation requires a sufficiently extended treatment to suggest a separate presentation [10]. To avoid unnecessary repetitions, also the detailed analysis of the Riemann surface associated with (81) is postponed to [10].

## 7. Outlook

In this paper we report a deeper analysis of the model introduced in [1] explaining many results that were reported there without proof (such as the derivation of the general solution by quadratures) while adding some new material (such as a detailed analysis of the equilibrium configurations, small oscillations and similarity solutions of the model). The novelty of this approach in accounting for a new phenomenology associated with chaotic motion in dynamical systems lies in the fact that the solution is a multivalued function of (complex) time, and a detailed analysis of its Riemann surface leads to very specific predictions in the simpler ( $\mu$  rational) cases, while it also unveils a source of irregular behaviour in the more complicated ( $\mu$  irrational) cases—unpredictable inasmuch as the determination of its evolution requires knowledge with arbitrarily large precision of the initial data. The full analysis of the dynamics of this model—including the geometry of the associated Riemann surface—is postponed to a future publication [10].

The purpose of this series of papers together with other related projects [15, 17] is to go beyond the local analysis performed in the literature relating analytic properties of solutions in complex time with dynamical properties of the model (Painlevé–Kowalewskaya and its non-meromorphic extensions) and to perform a full description of the *global* properties of the Riemann surface. This full description requires not just finding the type of branch points and their positions, but specifying how the different sheets of the Riemann surface are attached together at those branch points.

Whenever possible, such an approach provides very detailed information on the dynamics that cannot be obtained by the more classical local analyses.

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## Appendix A

In this appendix we solve, in the semisymmetrical case, see (2), the nonlinear algebraic equations (23) that characterize the equilibrium configurations and we thereby compute the ‘eigenvalue’  $\gamma^{(3)}$ , namely we obtain its two expressions (36a) and (36b).

The equations to be solved read (see (23))

$$\alpha_1 = \frac{2g}{\alpha_1 - \alpha_3} + \frac{2f}{\alpha_1 - \alpha_2}, \quad (\text{A.1a})$$

$$\alpha_2 = \frac{2g}{\alpha_2 - \alpha_3} - \frac{2f}{\alpha_1 - \alpha_2}, \quad (\text{A.1b})$$

$$\alpha_3 = \frac{2g}{\alpha_3 - \alpha_1} + \frac{2g}{\alpha_3 - \alpha_2}, \quad (\text{A.1c})$$

and they of course imply the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \quad (\text{A.2})$$

It is now convenient to set

$$S = \alpha_1 + \alpha_2, \quad D = \alpha_1 - \alpha_2, \quad (\text{A.3a})$$

entailing

$$\alpha_1 = \frac{S+D}{2}, \quad \alpha_2 = \frac{S-D}{2}, \quad \alpha_3 = -S. \quad (\text{A.3b})$$

From (the sum of) (A.1a) and (A.1b) we easily get

$$S(9S^2 - D^2) = 24gS, \quad (\text{A.4})$$

and from this we get two types of solutions. The *first* solution is characterized by  $S = 0$ , implying (see (A.3b) and (A.1a))

$$\alpha_3 = 0, \quad \alpha_1 = -\alpha_2 = \alpha, \quad \alpha^2 = f + 2g, \quad (\text{A.5})$$

entailing (via (22)) the solution (34) for the equilibrium configuration, as well as (via (24b) with (2)) the expressions

$$\beta_3 = \frac{f}{2(f+2g)}, \quad (\text{A.6a})$$

$$\beta_1 = \beta_2 = \frac{2g}{f+2g}, \quad (\text{A.6b})$$

hence, via (32), the first expression, (36a), for  $\gamma^{(3)}$ .

The *second* solution is characterized by

$$9S^2 - D^2 = 24g. \quad (\text{A.7})$$

We now subtract (A.1b) from (A.1a) and we thereby easily get

$$D^2 = \frac{-8gD^2}{9S^2 - D^2} + 4f, \quad (\text{A.8})$$

hence, via the preceding relation,

$$D^2 = 3f, \quad S^2 = \frac{f+8g}{3}. \quad (\text{A.9})$$

And via (24b) with (2) and (A.3b) this is easily seen to yield

$$\beta_1 + \beta_2 + \beta_3 = \frac{f+8g}{6g}, \quad (\text{A.10})$$

namely, via (32), the second expression, (36b), of  $\gamma^{(3)}$ . Note moreover that, *in both cases*, one gets the relation

$$(\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2 = 6(f+2g), \quad (\text{A.11})$$

as can be easily verified from (A.5) as well as from (A.3) with (A.9).

## Appendix B

In this appendix we explain how to integrate the ODE (72) in the general case when the three coupling constants  $g_n$  are *all* different, namely when the restriction (2) identifying the semisymmetrical case does *not* apply, and we also provide the solution of the ODE (72) in the two special cases (belonging to the semisymmetrical class characterized by the restriction (2)) the treatment of which had been omitted in section 5, and as well in another special case not belonging to the semisymmetrical class.



*B.1. Solution of equation (72) in the general case*

In this subsection of appendix B we indicate how the ODE (72) can be integrated in the general case when the three coupling constants  $g_n$  are *all* different. It is then convenient to set

$$V(\tau) = \tan[\theta(\tau)], \tag{B.1}$$

so that this ODE reads

$$\frac{V'V(V^2 - 3)}{(V^2 + 1)(AV^3 + CV^2 + AV + C - 2)} = \frac{1}{(\tau - \tau_1)} \tag{B.2}$$

with

$$A = \frac{\sqrt{3}(g_1 - g_2)}{2(g_1 + g_2 + g_3)}, \quad C = \frac{4g_1 + 4g_2 + g_3}{2(g_1 + g_2 + g_3)}. \tag{B.3}$$

To integrate this ODE we set

$$AV^3 + CV^2 + AV + C - 2 = A(V - V_1)(V - V_2)(V - V_3), \tag{B.4}$$

so that the *three* quantities  $V_n$  are the *three* roots of this polynomial of *third* degree in  $V$ . We then decompose the rational function of  $V$  appearing in the left hand side of the ODE (B.2) in simple fractions,

$$\frac{V(V^2 - 3)}{(V^2 + 1)(AV^3 + CV^2 + AV + C - 2)} = \sum_{j=1}^5 \frac{\mu_j}{V - V_j}, \tag{B.5}$$

where of course

$$V_4 = i, \quad V_5 = -i, \tag{B.6}$$

and the *five* quantities  $\mu_j$  are easily evaluated in terms of the five roots  $V_n$ :

$$\mu_j = V_j(-3 + V_j^2) \prod_{k=1, k \neq j}^5 (V_j - V_k)^{-1}. \tag{B.7}$$

The integration of the ODE (B.2) is now trivial (using (B.5)), and it yields (using (B.6)) the final formula

$$[V(\tau) - i]^{\mu_4} [V(\tau) + i]^{\mu_5} \prod_{n=1}^3 [V(\tau) - V_n]^{\mu_n} = K(\tau - \tau_1), \tag{B.8}$$

where  $K$  is the integration constant.

*B.2. Solution of equation (72) in two special subcases of the semisymmetrical case*

In this subsection of appendix B we provide the solution of the ODE (72) in the two special subcases (of the semisymmetrical case) the treatment of which had been omitted in section 5, and as well in another special case *not* belonging to the semisymmetrical class. If

$$g_1 + g_2 + g_3 = 0, \tag{B.9}$$

$\rho$  is constant (namely  $\tau$ -independent,  $\rho(\tau) = \rho(0)$ , see (69c)). Moreover, via the restriction (2) characterizing the semisymmetrical class, we get (see also (14))

$$f = -2g, \quad \mu = 0. \tag{B.10a}$$

Then (74) is replaced by

$$u(\tau) \exp[-2u^2(\tau)] = \exp\left[\frac{3f(\tau - \tau_0)}{\rho^2(0)}\right]. \quad (\text{B.10b})$$

Let us also note that, if (2) were replaced by

$$g_1 = -g_2 = g, \quad g_3 = 0, \quad (\text{B.11a})$$

which is also consistent with (B.9), then (74) with (71) would be replaced by

$$\theta(\tau) + \sin[2\theta(\tau)] = \frac{2\sqrt{3}g(\tau_0 - \tau)}{\rho^2(0)}. \quad (\text{B.11b})$$

Returning to the semisymmetrical case characterized by validity of the restriction (2) we now consider the second case whose treatment had been omitted in section 5, namely

$$f = -8g. \quad (\text{B.12})$$

Note that in this case  $\mu$  diverges, see (14). Then (74) is replaced by

$$u(\tau) \exp[u^2(\tau)] = [K(\tau - \tau_1)]^{-1/2}. \quad (\text{B.13})$$

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